## Calculus in Higher Dimensions

Background

This topic deals with extending concepts such as limits, continuity, differentiation, and integration, studied in first year calculus, to functions of several variables. Topics covered:

- Continuity of functions of several variables.

- Limits, partial derivatives, gradients, directional derivatives, divergence, and curl and apply these concepts to problem solving.

- Nature of extrema and optimization problems using Lagrange multipliers.

- Determine double and triple integrals and use them to calculate areas and volumes.

- Determine line, surface and flux integrals and apply the theorems of Green, Stokes and Gauss, which relate these types of integrals.

The Study Guide Splits the above topics into three distinct units:  
Basic Concepts

* Preliminaries (Sets, Relations, Implications, Symbols)
* N-dimensional Euclidian space (R, dot products, Norm, Distance, Unit Vectors, Basis Vectors, Angle between vectors, Cross Product, Lines, Subsets)
* Functions (visualisation, Rn-Rp)

Differentiation

* Limits and Continuity (R-R functions, Rn-R functions, Real Valued functions, Limits along curves, Vector Valued functions, Continuity)
* Derivatives Real Valued functions (One Variable)
* Derivatives Vector Valued functions (Chain Rule, Piecewise smooth curves)
* Derivatives Real Valued functions (Several Variables) (Rn-R functions, Gradient of Rn-R functions, Differentiability of Rn-R functions, Chain Rule, Directional Derivatives Rn-R functions, Potential Functions, Higher order Partial Derivatives)
* Derivatives of Vector Field
* Taylor Polynomials (R-R functions, Rn-R functions)

Integration

* Single Integrals
* Double Integrals
* Triple Integrals
* Line Integrals
* Surface Integrals
* Flux Integrals
* Theorems (Green, Gauss, Stokes)

**Lesson 1**

dimensional Euclidean Space

: One-dimensional space

Can be represented as a straight line

Corresponds with the set of real numbers

Written as the ordered tuple

: Two-dimensional space

Can be represented geometrically as a plane

Corresponds with two mutually perpendicular copies of , called the axis and axis

Origin denoted , is the point

Written as the ordered pair

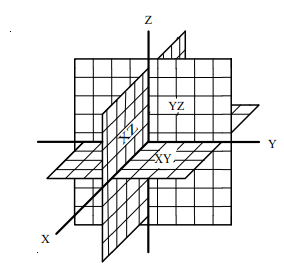
: Three-dimensional space

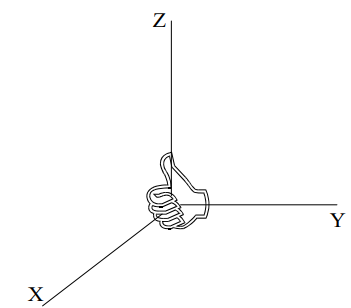
Can be represented geometrically as a plane

Corresponds with three mutually perpendicular copies of , called the axis, axis and axis

Origin denoted , is the point

Written as the ordered tuple

 *right hand rule coordinate planes in*



**Lesson 2**

Vectors in

The standard geometric definition of vector is as something which has direction and magnitude but not position. Since vectors have no position, we may place them wherever is convenient.

A vector in is a tuple

written as: and

**Scalar Multiplication**

Two vectors and are parallel if they are scalar

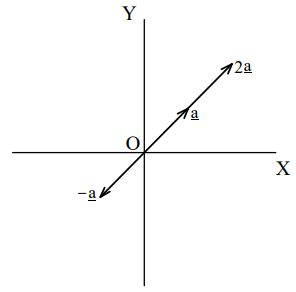
multiples of one another

where

*Algebraic definition of parallel vectors in*

Example: Which of the following vectors parallel?

*This number should not have a remainder*



**Cross Product**

Two vectors and are perpendicular

their scalar product is equal to zero

Example: a vector perpendicular to the

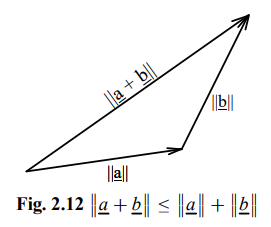
vectors and

**Dot Product**

The Norm in (length in )

*Dot product is a real number (scalar product)*

Example: the length of the vector



Perpendicular Vectors:

Graphical user interface, text, application

Description automatically generated

Parallel Vectors:

A picture containing text

Description automatically generated

OR

Graphical user interface, application

Description automatically generated

**How to tell if a point lies on a line**

**Graphical user interface, application

Description automatically generated**

**OR**

**Diagram, text

Description automatically generated**

**Lesson 3**

Planes in

A non-zero vector is parallel to a plane in if it is parallel to some line in the plane.

A non-zero vector is perpendicular to a plane if it is perpendicular to every line in the plane

A plane is uniquely determined by a point on the plane and a vector perpendicular to the plane. Any vector perpendicular to a given plane is called a normal of the plane

**Dot Product**

is a point on ,

is a vector perpendicular to

is any other point on

Thus, the plane consists of all points

in that satisfy the equation

Or

A picture containing photo, air, person, flying

Description automatically generated

Equation of a plane:

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**Lesson 4**

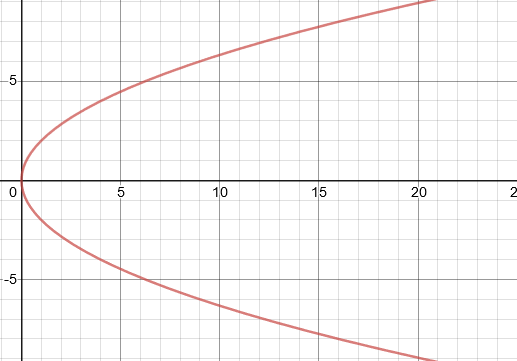
Parametric Equations

A parametric equation is where the x and y coordinates are both written in terms of another letter. This is called a parameter and is usually given the letter t or . ( is normally used when the parameter is an angle and is measured from the positive x-axis.)

Example: Plot the graph of

*Chose some random values for t*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |
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|  |  |  |  |  |  |  |  |  |



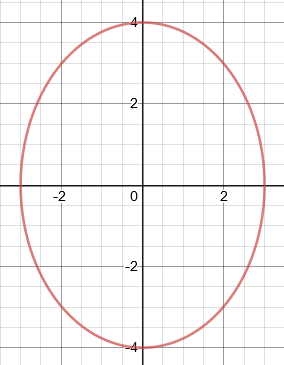
Desmos Graphing Calculator

*Don’t simplify further,*

Example: Plot the graph of

*Chose values for t which will give a good range of points for*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

*Try write the equation in the form ,*

Desmos Graphing Calculator

ASS1 Q1–:

Find the point of intersection , If there is one, of the following lines:

If and intersect, there is a point that lies on both lines. There must be such that:

*and are only distinguished for legibility*

If and intersect, there is a point that lies on both lines. There must be such that:

If and intersect, there is a point that lies on both lines. There must be such that:

Therefore, , which is a point on and

If and are in the plane they describe, the normal to the plane must be perpendicular to and

*Cross product*

*Find the determinant of , and*

*The sign of must be a minus*

*To evaluate a 2x2 matrix, use*

, which is a point on the plane

*Dot product*

Domain of multivariable function

<https://www.youtube.com/watch?v=CxtgMNAQr8g>

Basically, you are looking for points where the function is not defined.

(Natural logs, Square root, Piecewise functions etc)

ASS1 Q2: Find the equations for the line of intersection of two planes

*Choose the arbitrary point where*

*Use matrix to solve system of equations*

*Multiply both sides of the matrix equations with the inverse*

Therefore, a point that passes through the plane is

Therefore,a vector that describes the direction which it travels is

Therefore, the equation for the line of intersection of two planes

ASS1 Q2: Find the equations for the line of intersection of two planes

**Alternate Solution**

*Convert into matrices*

**Lesson 6**

Quadric Surfaces

<https://tutorial.math.lamar.edu/classes/calciii/quadricsurfaces.aspx>

These are the three-dimensional representation of a surface in 3 dimensions. These are in the explicit form where is a quadratic form in

Quadric surfaces are the graphs of any equation that can be put into the general form

|  |  |  |  |
| --- | --- | --- | --- |
| Shape | Form | Shifted form |  |
| A picture containing object, umbrella  Description automatically generated  Ellipsoid |  |  | If then we have a sphere |
| A close up of a wire fence  Description automatically generated  Cone |  |  |  |
| A picture containing building  Description automatically generated  Cylinder |  |  | This is a cylinder whose cross section is an ellipse  If then we have a cylinder whose cross section is a circle |
| A picture containing game, basketball  Description automatically generated  Hyperboloid (of one sheet) |  |  | The variable with the negative in front of it will give the axis along which the graph is centred |
| A picture containing game, table  Description automatically generatedHyperboloid (of two sheets) |  |  | The variable with the positive in front of it will give the axis along which the graph is centred. |
| A close up of a basket  Description automatically generatedElliptic  Parabaloid |  |  | As with cylinders this has a cross section of an ellipse and if it will have a cross section of a circle |
| A picture containing dress, umbrella  Description automatically generated  Hyperbollic Hyperboloid  A picture containing drawing  Description automatically generated  A picture containing building, tower  Description automatically generated |  |  |  |

ASS1 Q3:

a) Consider the surfaces in defined by the equations

A picture containing racquetball, game

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is a Cone (top portion)

is an Elliptic/Circular Paraboloid

is a Hyperbolic Hyperboloid) Intersection where

Desmos Graphing Calculator

*The intersection of and is a circle*

Sphere

*, the ellipsis does not change size up and down z*

Step 1: let ,

*is a ellipsis in x-y*

Step 2: let ,

*is an ellipsis in y-z*

Step 3: let ,

*is an ellipsis in x-z*

*In the above example, it’s just made a full circle for convenience*

Ellipsoid

*shifts down along x-axis*

*shifts down along y-axis*

*shifts down along z-axis*

Step 1: let

*is a ellipsis in x-y*

*In the above example, it’s just made a full ellipsis for convenience*

Step 2: let ,

*is an ellipsis in y-z*

Step 3: let ,

*is an ellipsis in x-z*

Cylinder

Step 1: draw an ellipsis

*is a ellipsis in x-y*

*the ellipsis does not change size up and down z*

**Graphing Quadric Surfaces**

Paraboloid

*shifts down along x-axis*

*shifts down along y-axis*

Step 1: let

*is a parabola in z-y*

*(one linear term and one square term is a parabola)*

Step 2: let ,

*is an ellipsis that changes size along z*

Cone

*shifts down along x-axis*

*shifts down along y-axis*

Step 1: let

*is a absolute value function (V-shape)*

Step 2: let ,

*is an ellipsis that changes size along z*

Hyperboloid of One Sheet

*on x-axis*

*on y-axis*

*on z-axis*

*shifts down along x-axis*

*shifts down along y-axis*

Step 1: let

*is a horizontal hyperbola in z-y*

Step 2: let ,

*is an ellipsis that changes size along z*

Hyperboloid of Two Sheets

*, on x-axis*

*, on y-axis*

*, on z-axis*

*shifts down along x-axis*

*shifts down along y-axis*

Step 1: let

*is a vertical hyperbola in z-y*

Step 2: let ,

*is an ellipsis that changes size along z*

Hyperbolic Paraboloid

*Looks like a paraboloid but it is hyperbolic*

Example: Draw a saddle

*Stationary point (0,0)*

Step 1: let

*is a parabola in z-y*

*(one linear term and one square term is a parabola)*

Step 2: let ,

*is a parabola in z-x*

*(one linear term and one square term is a parabola)*

Step 3: let ,

*is in x-y*

Step 4: Look at k value, construct many hyperbolas

**Lesson 7**

Partial Differentiation

Differentiation you have dealt with until now has been with functions over a single variable

But realistically, you will come across functions with two or more independent variables, which may represent space/time.

Like quadric surfaces, these can be explicitly represented in the form

. If , then:

or or or or

or or or or

Example: Given , find and

1) Find

2) Find (partial derivative with respect to )

­*treat y and z like constants*

2) Find (partial derivative with respect to )

­*treat x and z like constants*

3) Find (partial derivative with respect to )

­*treat x and y like constants*

4) Find

4) Find

Example: March 2010 Q6

or

Critical/Stationary points:

*Use the value of x for critical points*

*Use the value of y for critical points and*

and

Second Order Derivatives:

*Differentiate with respect to*

OR

*Differentiate with respect to*

Saddle points:

which is a saddle point

which is not a saddle point

Therefore, is a saddle point

**Lesson 8**

Gradient Vectors and maximum rate of change

<https://www.youtube.com/watch?v=xBKhPZ5RgzQ>

This topic deals with finding the gradient of a function. This site explains this concept in a summary:

<http://sites.science.oregonstate.edu/math/home/programs/undergrad/CalculusQuestStudyGuides/vcalc/grad/grad.html>

The gradient of a function is the 2D vector function:

The maximum rate of change at a given point:

This is useful in cases where you want to transform a 3D interpretation of a graph to a 2D one (i.e. Quadric surface of a mountain to a contour plot)

This also helps us find the direction that the gradient vector increases towards the fastest

. ­

*the gradient vector increases the fastest towards y*

Gradient of a function

Text

Description automatically generated

**Two Variables**

Consider the function defined by

Find an equation for the tangent plane to the graph at

**Wolfram Alpha**

tangent plane equation

curve function: x^2y+2

point: {1,2}

Z = 4x + y - 2

*This is equivalent to (4,1)*

Therefore

Therefore

Therefore, the equation for

**Three Variables**

Consider the function defined by

Find an equation for the tangent plane to the graph at

**Wolfram Alpha**

tangent plane equation

curve function: yz+2xz-xy

point: {1,0,2}

W = 4x + y +2z -4

*This is equivalent to (4,1,2)*

Therefore

Therefore

Therefore, the equation for

Equation for that is perpendicular to

General Chain Rule: Multivariable functions

Consider the functions and below. Use the chain rule to determine

Compute:

Method of Lagrange:

Example: ASS 2 SEM 2 Qc

Let the length be x

Let the width be y

Let the height be z

Let the volume of the function be the objective function:

Therefore, the restriction as a function:

Gradients:

Optimum of function at:

Write all variables in terms of :

Write all variables in terms of :

Substitute variables back into restriction:

Maximum volume (write in English):

Example: JAN 2010 Q7

Let the length/width be x

*The base is square so x and y will be equal*

Let the height be y

Let the volume function be:

Let the constraint be:

Method of Lagrange:

Write in terms of :

Substitute back into constraint:

Maximize volume function:

Therefore, the maximum volume of the rectangular block is

Method of Lagrange:

Example: OCT 2010 Q7

Let the radius be r

Let the height be h

Let the volume of the function be the objective function:

Therefore, the constraint as a function:

*surface area of a cylinder*

Gradients:

Optimum of function at:

Write all variables in terms of :

TODO

Write all variables in terms of :

Substitute variables back into restriction:

Maximum volume (write in English):

**Lesson 9**

Cartesian equation for the line `

<https://www.s-cool.co.uk/a-level/maths/vectors-lines-and-planes/revise-it/straight-lines>

**Lesson 9**

Limits for multi-variable functions

Limits help solve the problem of indeterminate form

Calculating instantaneous velocity is an example of a limit

Say we have two different limits:

Then all the following are true:

*Product and sum rules*

Example:

Since this function has concepts we are familiar with (quotient, addition, product), we can just plug in values for and

Example:

try factoring this time

**Lesson 10**

Limits for multi-variable functions: Precise definition

<https://www.youtube.com/watch?v=CCDxj3g_BVk>

Factoring and plugging in a value for and is easy, but what about the definition?

A close up of a blackboard

Description automatically generated

**Epsilon-Delta definition**

Single variable

If

Then

*The distance needs to be positive, hence ABS*

*As x approaches some random point a, the limit is L*

With the above graph we are looking to define intervals around .

We do this so that we can ignore everything outside of the interval

This helps us use the precise definition

on the y axis corresponds with

on the y axis corresponds with

If we chose a good value for (a delta distance from ), automatically we have a corresponding good value for (an epsilon distance from )

A close up of a blackboard

Description automatically generated

**Epsilon-Delta definition**

Multi variable

If

Then

*Because . This will always be a positive number*

*As the coordinate system approaches some random coordinate point the limit is L*

With the above graph we are looking to transfer everything we now know about a single variable onto a 3-dimensional system.

So instead of two lines, we now have a circle as our interval

is now the radius of this circle

A value of will correspond to some value of

How to use the precise definition

[1] Continuous: Plug in values or factorise

Path 1

*Path along x-axis, as we approach origin*

[2] Not Continuous: Show that limit (try different paths)

*In single variable, this was just showing that*

*In multi variable, test several different paths. Two of these need to be different*

Path 4

*Parabola path that passes through origin*

Path 3

*Non-vertical line that passes through origin.*

Path 2

*Path along y-axis, as we approach origin*

Path 5

*Parabola path that passes through origin*

[3] Use precise definition

*If you get the same answer for all random paths, use the precise definition*

*At this point, we think is (all our paths led to ). Also and*

If

Then

*Neaten the above*

If

Then

whenever

*We start by removing ABS. The denominator doesn’t need ABS as sqrt b positive*

*The same applies for , it will be positive. We don’t need the zero inequality*

whenever

*Our goal is now to find a relationship between delta and epsilon.*

Let’s create some inequalities

*If you try for instance, RHS will be greater*

*Since the numerator is greater than the denominator.*

*1 is used because*

*Since the LHS < 1 (e.g. ), it will probably reduce y*

*e.g. -3 (LHS) will be less than 3 (RHS)*

Therefore,

SEM 1 ASS 1: Q4

Given ,

**Epsilon-Delta definition**

Multi variable

If

Then

*Because . This will always be a positive number*

*As the coordinate system approaches some random coordinate point the limit is L*

Prove from first principles that

If

Then

*Substitute =-1, ,*

If

Then

*Substitute , .*

*Add and for extra constants created by substitution*

If

Then

If

Then

Now we can start with the calculation using the function

From the Epsilon-Delta definition above, we now must find some relationship between and

or

*triangle inequality*

*Substitute ,*

From our earlier definition

If

Then

But for any if , then

Therefore

*Substitute*

If

Then

*Multiply all expressions by 6*

If

Then

If

Then

We can see that for any . Therefore

**Lesson 11**

Parameterizing a curve

A curve is defined as:

[1] Simple

Changes direction but does not cross itself while changing direction. A simple curve can be open and closed both.

[2] Smooth

Has derivative at every point.

[3] Piecewise Smooth

Is made up of finite number of connected smooth curves

[4] Open or Closed

Open: has distinct endpoints

Closed: the endpoints coincide

**Lesson 14**

Taylor Polynomials functions

This helps us obtain accurate approximations can be obtained by using higher order derivatives.

Example: Find the 4th degree Taylor polynomial for centered at and use it to approximate

*General formula for Taylor polynomial*

*Substitute into general formula*

*Therefore, approximation of*

Example: Find the 4th degree Taylor polynomial for and use it to approximate

*General formula for taylor polynomial where*

*Substitute into general formula*

*Therefore, approximation of*

**Lesson 15**

Taylor Polynomials functions

<https://math.libretexts.org/Bookshelves/Calculus/Supplemental_Modules_(Calculus)/Multivariable_Calculus/3%3A_Topics_in_Partial_Derivatives/Taylor__Polynomials_of_Functions_of_Two_Variables>

<https://math.stackexchange.com/questions/1188065/multivariable-taylor-polynomial>

Very similar to functions. To calculate these, you need to calculate degree polynomials in ascending order. Start at the 1st degree, then 2nd, then 3rd, then nth

*General formula for1st degree Taylor polynomial*

*General formula for 2nd degree Taylor polynomial*

*Or*

*General formula for 3rd degree Taylor polynomial*

Example: find the 3rd order Taylor polynomial about the point of the function

<https://mathinsight.org/taylor_polynomial_multivariable_examples>

First, second and third order derivatives

Evaluate derivatives at the point

First Order:

**2D vector**

*General formula for1st degree Taylor polynomial*

Second Order:

*General formula for 2nd degree Taylor polynomial*

**2x2 symmetric matrix**

***Hessian Matrix***

Third Order:

[*https://math.stackexchange.com/questions/1779418/taylor-series-third-order-approximation/1779506*](https://math.stackexchange.com/questions/1779418/taylor-series-third-order-approximation/1779506)

A close up of a clock

Description automatically generated

*General formula for 3rd degree Taylor polynomial*

**2x2x2 symmetric tensor**

Or

*General formula for 3rd degree Taylor polynomial*

**Lesson 16**

Jacobian matrix

<https://www.wolframalpha.com/widgets/view.jsp?id=23b545022f0fb5a2ca68d0bac5084191>

is a Conservative vector field when:

Second order derivatives of are continuous

*Could be smooth i.e. have derivatives at every point*

scalar curl of is zero

is defined on all

ASS 1 SEM 1 Q11:

Consider the 3-dimensional vector field

Write down the Jacobian matrix

*Careful with inputting in wolfram alpha, it’s buggy*

**Wolfram Alpha**

jacobian of (x^2+y^2z, 2.zxy+z^3, xy^2+3z^2y)

A screenshot of a cell phone

Description automatically generated

Example: ASS1 Qk ii)

**Wolfram Alpha**

jacobian of (2.zxy, x^2.z+2.yz^2,x^2.y+2.y^2.z+e^z)

Example: ASS1 Qk iii)

**Wolfram Alpha**

curl [2.zxy, x^2.z+2.yz^2,x^2.y+2.y^2.z+e^z]

OR

**Lesson 17**

Critical Points

**Wolfram**

stationary points 1/y-1/x-4x+y

**Lesson 18**

Parabolas

The simplest form of a parabola is

Parabolas of the form

A parabola is also a conic section

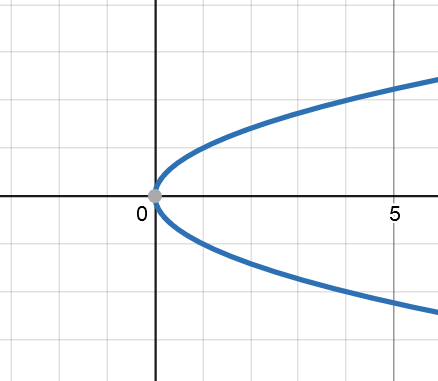
A parabola is a curve where any point is at an equal distance from:

* A fixed point called the focus
* A fixed straight line called the directrix

The lowest point is called the vertex. It is halfway between the focus and directrix

Chart, line chart

Description automatically generated



: parabola opens upward

: parabola opens downward

: focus point ( units away from origin)

: directrix (graph centred at origin)

: ( units away from origin)

: parabola opens rightward

: parabola opens leftward

: focus point ( units away from origin)

: directrix (graph centred at origin)

: ( units away from origin)

**Lesson 18**

Parametric Equations of Parabolas

<https://www.math-only-math.com/parametric-equations-of-a-parabola.html>

**TLDR**: The best and easiest form to represent the co-ordinates of any point on the parabola:

: : parabola opens upward

: : parabola opens downward

Or

: : parabola opens rightward

: : parabola opens rightward

Example: write the parametric equation of the parabola

*This is already of the form*

Therefore, the parametric equations are:

:

and

SEM2 ASS2: write the parametric equation of the parabola

*This is already of the form*

*will not go below*

Therefore, the parametric equations are:

:

and

SEM1 ASS2: write the parametric equation of the parabola

*the form*

*will not go below*

Therefore, the parametric equations are:

:

and

**Lesson 19**

Optimization

[*https://socratic.org/questions/how-do-you-use-lagrange-multipliers-to-find-the-volume-of-the-largest-rectangula*](https://socratic.org/questions/how-do-you-use-lagrange-multipliers-to-find-the-volume-of-the-largest-rectangula)

**Lesson 20**

Regions

Type region is if there is at least an obvious upper and lower bound:

Vertical slices for the area

*Make the subject of the equation*

A close up of a necklace

Description automatically generated

Type region is if there is at least an obvious right and left bound:

Horizontal slices for the area

*Make the subject of the equation*